is nearly constant. The decay of peak tangential velocities given in Eq. (4) is substantiated by empirical results.

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A Closed-Form Solution to Oblique Shock-Wave Properties

Vincent R. Mascitti*

NASA Langley Research Center, Hampton, Va.

THIS Note is concerned with the direct computation of oblique shock-wave properties with freestream Mach number and flow-deflection angle as the independent variables.

The equations governing oblique shock relations may be written in the form

$$\sin^6\theta + b\sin^4\theta + c\sin^2\theta + d = 0$$

where

$$b = -\left[\frac{M_1^2 + 2}{M_1^2}\right] - \gamma\sin^2\delta$$

$$c = (2M_1^2 + 1)/M_1^4 + [(\gamma + 1)^2/4 + (\gamma - 1)/M_1^2]\sin^2\delta$$

$$d = -\cos^2\delta/M_1^4$$

and

θ = shock-wave angle
M_1 = freestream Mach number
δ = deflection angle
γ = ratio of specific heats

which is cubic in $\sin^2\theta$, having three real roots, the smallest of which results in a decrease in entropy.

Contrary to the statement of Ref. 1, that no convenient explicit relation exists for this case, there is indeed a general solution for a cubic. The mathematical derivation can be found in Ref. 2. From Ref. 2, the solution for a cubic having three real roots is

$$\sin^2\theta = -b/3 + \tfrac{2}{3}(b^2 - 3c)^{1/2}\cos[(\phi + n\pi)/3]$$

where

$$\cos\phi = (\tfrac{9}{2}bc - b^3 - \tfrac{27}{2}d)/(b^2 - 3c)^{3/2}$$

and $n = 0$ corresponds to the strong shock solution; $n = 2$ results in a decrease in entropy; and $n = 4$ corresponds to the weak shock solution. Although many readers may be aware of this solution, the wide use of iteration schemes to solve this problem has prompted the author to set down the explicit solution in general terms.

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* Aerospace Engineer, Advanced Configuration Group.

Free Vibration of Simply Supported Parallelogrammic Plates

Somayajulu Durvasula*

Indian Institute of Science, Bangalore, India

Nomenclature

a,b = dimensions of the plate, see Fig. 1a
D = plate rigidity, $Eh^3/12(1 - \nu^2)$
E = Young's modulus of the material of the plate
h = plate thickness
k = frequency parameter, $(\rho h/D)^{1/2}\omega a^2/\pi^2$
k_m = frequency parameter of membrane, $(\mu/S)^{1/2}\omega a/\pi$
m,n = number of half sine waves in the two directions x_1 and y_1, respectively
S = uniform tension per unit length of stretched membrane
x,y = rectangular coordinate system defined in Fig. 1a
x_1,y_1 = oblique coordinates defined in Fig. 1a
ρ = mass density of the plate material
ψ = angle of skew, defined in Fig. 1a
ω = frequency of oscillation in rad/sec
μ = mass per unit area of membrane
ν = Poisson's ratio

Introduction

IN this Note, the results of numerical calculations for the first few frequencies of simply supported parallelogrammic plates, using the Rayleigh-Ritz method employing double Fourier sine series in oblique coordinates, are presented. Interesting features, hitherto unreported in the literature, such as 1) the skew angle splitting the degenerate frequencies of rectangular plates to distinct ones and 2) the "frequency crossing" of the modes of simply supported skew plates, are discussed. In fact, it has been shown in Ref. 1 that these features are also exhibited by clamped skew plates.

The literature does not contain adequate results for the frequencies of simply supported parallelogrammic plates. Conway and Farnham[2] calculated only the fundamental frequency for a few configurations of the plate by point matching, using a mathematical relationship that exists between the problems of a simply supported polygonal plate and a polygonal membrane of the same geometry.[3–5] This relationship shows that the eigenvalues of the plate are squares of the eigenvalues of the membrane, whereas the eigenfunctions are identical. Weinstein[6] reports the upper and lower bounds of the frequencies of modes symmetric about both the diagonals of rhombic membrane, which have been calculated by Stadter[7] in an unpublished report. These values serve admirably for comparison with the results of simply supported rhombic plates on the basis of the aforementioned relationship.

Details of Solution

The vibration problem becomes a particular case of panel-flutter problem of simply supported parallelogrammic panels, which is discussed in detail in Ref. 8. Consequently, the

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* Assistant Professor, Department of Aeronautical Engineering.

Table 1 Comparison of the frequencies $k = (\rho h/D)^{1/2}\omega a^2/\pi^2$ of rhombic simply supported plate (for modes which are symmetric about both the diagonals)

Mode no.	Authors	Nature of solution	Skew angle ψ 15°	30°	45°
1	Stadter[a]	l.b.[b]	2.1137	2.5210	3.5170
		u.b.[c]	2.1147	2.5245	3.5287
	Conway and Farnham[d]	R[e]	...	2.519	...
		P[f]	2.119	2.402	3.234
	Present Note	u.b.	2.133	2.630	3.926
2	Stadter	l.b.	7.9960	8.4807	10.143
		u.b.	8.0126	8.5061	10.191
	Present Note	u.b.	8.078	8.739	10.90
3[g]	Stadter	l.b.	11.018	14.186	18.659
		u.b.	11.038	14.274	18.941
	Present Note	u.b.	11.08	14.59	20.42

[a] Stadter's results are as quoted in Ref. 6.
[b] l.b. ~ lower bound.
[c] u.b. ~ upper bound.
[d] Ref. 2.
[e] R ~ Rhombic plate analysis.
[f] P ~ Parallelogram plate analysis.
[g] Mode 3 of the doubly symmetric group corresponds to mode 6 for $\psi = 15°$, mode 7 for $\psi = 30°$, and 45° of the complete spectrum.

mathematical development is left out for the sake of brevity and only the final results are presented in this Note. The resulting matrix equation of the vibration problem splits into two sets, the even (E) set and the odd (O) set, representing modes that are skew symmetric and skew antisymmetric, respectively.[9] The eigenvalues and eigenvectors of the two separate matrix equations of half the order are calculated, and the complete spectrum is obtained by considering the two spectra together and arranging them in the ascending sequence.

Results and Discussion

Numerical calculations for the first few frequencies have been made for plates with side-ratio a/b equal to 1, $\frac{2}{3}$, $\frac{1}{2}$ and $\frac{1}{3}$ and with ψ varying from 0° to 50°. In a majority of the cases, the number of terms was taken up to $M = 6, N = 6$; this results in matrices of order 18 × 18 each for both the even and the odd cases. All the values are given in terms of the parameter k. The tables giving these results being lengthy are not reported in this Note; they may be found in Ref. 9.

A detailed examination of the convergence is made. In general, it is found that convergence becomes slow for ψ = 45°, as may be expected; further, it tended to be the slowest for $a/b = 1$. The convergence of the first four modes for $a/b = 1$ and $\frac{1}{2}$, and of the first five modes for $a/b = \frac{1}{3}$ is essentially satisfactory up to a skew angle of 45°. Accordingly, the results for only these modes have been presented in the form of graphs showing the variation of the frequencies with the skew angle. Where the convergence is felt to be slightly less satisfactory, the curves have been shown in dotted lines. An assessment of the accuracy is possible by a comparison with Stadter's results for membranes, and using the plate-membrane relationship.

Stadter's results (as quoted in Ref. 6) are only for modes which are symmetric about both the diagonals, whereas the present calculations include all the four symmetry groups of the rhombus. Thus, a given mode (other than the fundamental) of the doubly symmetric group corresponds to a higher mode of the complete spectrum in which the modes have all been numbered in the ascending order of their frequencies.

In Table 1, the values of the frequency parameter k for the rhombic simply supported plate are given along with the squares (k_m^2) of the frequency parameter of rhombic membrane obtained by Stadter. The fundamental frequency calculated by Conway and Farnham using rhombic plate and parallelogrammic plate analyses are also given. The third mode of the symmetric-symmetric group of Stadter corresponds to the sixth mode for $\psi = 15°$ and the seventh mode for $\psi = 30°$ and 45° of the complete spectrum. It is seen that the maximum discrepancy for a skew angle of 30° is of the order of 4%. For a skew angle of 45°, it is of the order of 11%. While the point-matching results of Conway and Farnham[2] for the fundamental frequency alone of rhombic plates by the rhombic-plate analysis compare favorably, the results obtained by the parallelogram-plate analysis are not as accurate. The values of the fundamental frequencies for parallelogrammic plates by the latter analysis tend to be even less accurate.

The variation of the natural frequencies with the skew angle are shown in Figs. 1a–1c. The mode number and the group even (E) or odd (O) to which it belongs are also indicated. In the inset of the graphs, the modes which are crossing, the skew angle at crossing (obtained by linear interpolation), and the corresponding k are all given.

In Fig. 1a for $a/b = 1$, the modes labelled 2 and 3 are the degenerate modes ($m = 2, n = 1$; $m = 1, n = 2$) having the same frequency $k = 5$ in the case of the square plate ($\psi = 0°$). It is interesting to notice that with even a small angle of skew, this degeneracy (or multiplicity) disappears and they become modes with distinct frequencies. These modes belong to the odd group, i.e., they are skew antisymmetric. Similarly, the skew angle splits the degeneracies of the modes 5, 6, 7, and 8 (not shown in the graph).

The second interesting feature is the frequency crossing. One notices that the order of the modes 3, 4 belonging to the odd and even groups, respectively, gets interchanged. This occurs at a skew angle of 40.5°. Similar crossings take place between higher modes. Such crossings of frequency curves, representing the variation of frequency with side-ratio, of modes belonging to different symmetry groups for rectangular clamped plates[10] and cantilever plates[11] exist. The crossing of curves showing the variation of frequency with the angle of skew of the modes of skew cantilever plates are known also.[12,13]

Figure 1b shows the variation of the frequencies of the first four modes for $a/b = \frac{2}{3}$. Modes 3 and 4 cross at ψ = 22.67°. Figure 1c shows the variation of the frequencies of the first four modes for $a/b = \frac{1}{2}$. Again, modes 5 and 6 (not shown) which are degenerate for $\psi = 0°$ are split into two distinct ones by the skew angle. Thus, it is seen that the introduction of a slight asymmetry into a system with inherent symmetry removes the degeneracies possessed by that

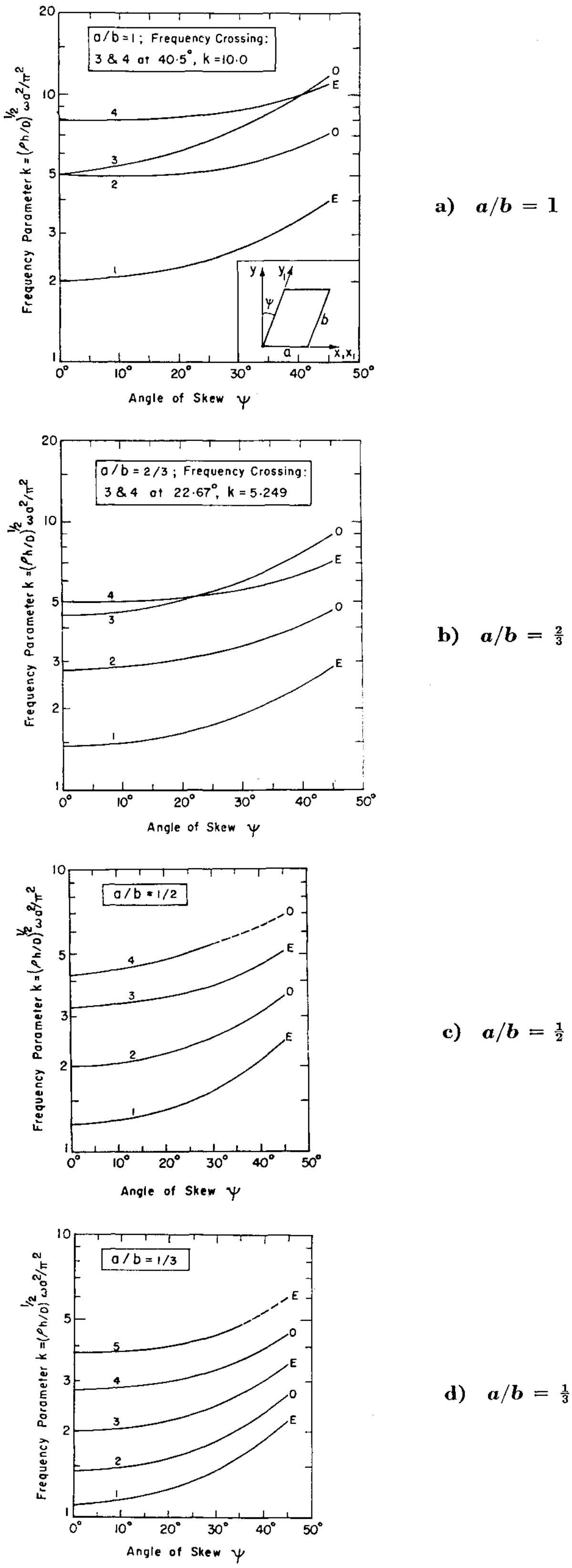

Fig. 1 Variation of natural frequencies with angle of skew.

system. Figure 1d shows the variation of the frequencies of of the first five modes for $a/b = \frac{1}{3}$.

It is interesting to observe that, while the skew angle tends to split the degeneracies of the modes of rectangular plates on one hand, it does bring about other degeneracies between different modes at certain other values corresponding to the frequency crossings. That is to say, the frequency crossings are degeneracies, since two different modes have the same frequency. Further, one observes that the frequency crossings are always between a pair of modes belonging to the opposite groups, even and odd.

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Maximum Stress of a Stiffened Circular Cylinder under Bending

TSAI-CHEN SOONG*
The Boeing Company, Seattle, Wash.

Nomenclature

A_s = cross-sectional area of the stringer, in.^{2}
d = stringer spacing, in.
E = Young's modulus of the skin, psi
h = distance from the neutral axis to the center of the cylinder, in.
M = bending moment, lb.-in.
R = radius of the circular cylinder, in.
t = skin thickness, in.

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* Research Specialist A, Stress Analysis Research, Structures Staff, Commercial Airplane Division.